Lecture 2

6.2* - The Natural Logarithm

In Calc I you learned that

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

as long as $n \neq -1$. But, what happens if $n = -1$?

Def: We define the natural logarithm by

$$\ln x = \int_1^x \frac{1}{t} \, dt$$

By construction, $\ln x$ is an anti-derivative for $\frac{1}{x}$.

Note: $\ln x$ is the area under $y = \frac{1}{t}$ between $t=1$ & $t=x$ for $x \geq 1$, and negative of the area under $y = \frac{1}{t}$ between $t=x$ and $t=1$ for $x < 1$.

Upon defining a new function, it's good to see what properties it has:

Properties of ln x

1) $D(\ln x) = (0, \infty)$, by construction

2) $R(\ln x) = (-\infty, \infty)$ We'll show this later

3)

x	>1	$=1$	<1
$\ln x$	>0	$=0$	<0

Follows from $\frac{1}{t} > 0$ for $t > 0$ and the order of the bounds in $\ln x = \int_1^x \frac{1}{t}\,dt$.

4) $\frac{d}{dx}(\ln x) = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$

Fundamental Theorem of Calculus

5) The graph of $y = \ln x$ is increasing, continuous, and concave down on $(0, \infty)$. $y = \ln x \Rightarrow y' = \frac{1}{x} \Rightarrow y'' = \frac{-1}{x^2}$.

$\ln x$ is differentiable $\Rightarrow$ it is continuous.

$y' > 0 \Rightarrow \ln x$ increasing & $y'' < 0 \Rightarrow \ln x$ concave down.

6) The function $f(x) = \ln(x)$ is one-to-one.

Since $f'(x) = \frac{1}{x}$ is always positive on $(0, \infty)$, $f(x)$ is strictly increasing, and thus is one-to-one.

7) There exists a unique number, e, such that $\ln e = 1$.

$\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0$. Using a Riemann sum with 3 rectangles and right endpoints, we get $\ln(4) > \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1$. So, by the intermediate value theorem, since $\ln x$ is continuous, there is a number e such that $\ln e = 1$. Since $\ln x$ is one-to-one, it is unique.

Algebraic Properties of ln

i) $\ln 1 = 0$ ii) $\ln(ab) = \ln a + \ln b$

iii) $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$ iv) $\ln a^r = r \ln a$

Proof

ii) Fix a constant $a > 0$. Let $f(x) = \ln x$ & $g(x) = \ln(ax)$ for $x > 0$.

$f'(x) = \frac{1}{x}$ & $g'(x) = \frac{1}{ax} \cdot a = \frac{1}{x}$.

So $f'(x) = g'(x) \Rightarrow f(x) + C = g(x)$. Plug in $x = 1$:

$$\ln 1 + C = \ln a \Rightarrow C = \ln a$$

$\Rightarrow \ln x + \ln a = \ln ax$. Letting $x = b$ gives $\ln ab = \ln a + \ln b$.

iv) $\frac{d}{dx}(\ln(x^r)) = \frac{1}{x^r} \cdot (rx^{r-1}) = \frac{r}{x}$ & $\frac{d}{dx}(r \ln x) = \frac{r}{x}$

So $\ln(x^r) = r \ln x + C$. Plug in $x = 1$ to see $C = 0$.

Finally, let $x = a$ to get $\ln a^r = r \ln a$

Ex: Expand $\ln \dfrac{x^2\sqrt{x^2+1}}{x^3}$

$$\ln \frac{x^2\sqrt{x^2+1}}{x^3} = \ln \frac{x^2(x^2+1)^{1/2}}{x^3} = \ln\left(x^2(x^2+1)^{1/2}\right) - \ln x^3$$

$$= \ln x^2 + \ln (x^2+1)^{1/2} - \ln x^3 = 2\ln x + \tfrac{1}{2}\ln(x^2+1) - 3\ln x$$

Ex: Combine into a single logarithm

$$\ln x + 2\ln(x+1) - \tfrac{1}{3}\ln(x-1)$$

$$= \ln x + \ln(x+1)^2 - \ln(x-1)^{1/3}$$

$$= \ln\left(x(x+1)^2\right) - \ln\sqrt[3]{x-1} = \ln \frac{x(x+1)^2}{\sqrt[3]{x-1}}$$

Ex: Evaluate $\displaystyle\int_1^{e^7} \frac{1}{t}\,dt$.

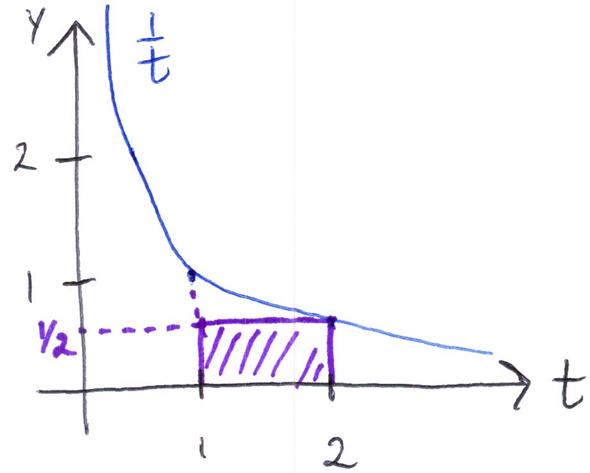
$$\int_1^{e^7} \frac{1}{t}\,dt = \ln e^7 = 7\ln e = 7\cdot 1 = 7$$

What are the values of:

$$\lim_{x\to\infty} \ln x \quad \& \quad \lim_{x\to 0} \ln x \ ?$$

Let's start with approximating $\ln 2$:

Using a Riemann sum with one rectangle and right endpoints, we get a lower estimate since $\frac{1}{t}$ is decreasing:



and so we get $\ln 2 > \frac{1}{2}(2-1) = \frac{1}{2}$.

Using properties of ln, we have then

$$\ln 2^n = n \ln 2 > \frac{n}{2}$$

So, if $x > 2^n$

$$\ln x > \ln 2^n > \frac{n}{2}$$

and this implies

$$\lim_{x\to\infty} \ln x = \infty$$

since $\frac{n}{2} \to \infty$ as $n \to \infty$

Likewise, $\ln \frac{1}{2^n} = -n \ln 2 < \frac{-n}{2}$. So, for $x < \frac{1}{2^n}$,

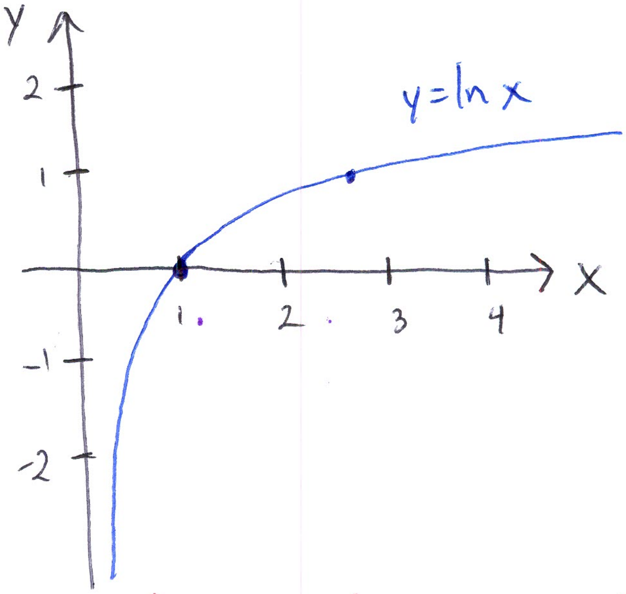
$$\ln x < \ln \frac{1}{2^n} < -\frac{n}{2}$$

and this implies

$$\lim_{x \to 0} \ln x = -\infty$$

since $\frac{-n}{2} \to -\infty$ as $n \to \infty$

Using what we now know, we can sketch the graph of $y = \ln x$:



typo in notes... missing ln.

Ex: Compute $\lim_{x \to \infty} \ln\left(\frac{x}{x^2-1}\right)$

Since $\frac{x}{x^2-1} \to 0$ as $x \to \infty$, because ln is continuous we have

$$\lim_{x \to \infty} \ln\left(\frac{x}{x^2-1}\right) = \lim_{y \to 0} \ln(y) = -\infty$$

A useful extension of ln (especially in differential equations) is obtained by composing it with the absolute value function:

$$\ln|x| = \begin{cases} \ln x, & x>0 \\ \ln(-x), & x<0 \end{cases}$$

It has the benefit of having a much larger domain than $\ln x$. Moreover, it is still an antiderivative of $\frac{1}{x}$, but now on the whole domain of it!

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x} \quad \& \quad \int \frac{1}{x}\,dx = \ln|x| + C$$

Using the chain rule we get:

$$\frac{d}{dx}\left(\ln|g(x)|\right) = \frac{1}{g(x)} \cdot g'(x) = \frac{g'(x)}{g(x)}$$

and by u-substitution:

$$\int \frac{g'(x)}{g(x)}\,dx \overset{u=g(x)}{=\!=} \int \frac{1}{u}\,du = \ln|u| + C = \ln|g(x)| + C$$

Ex: Find $\frac{d}{dx}\left(\ln\left|\sqrt[3]{x-1}\right|\right)$

$$\frac{d}{dx}\left(\ln\left|(x-1)^{1/3}\right|\right) = \frac{1}{(x-1)^{1/3}} \cdot \left(\frac{1}{3}(x-1)^{-2/3}\right) = \frac{1}{3} \cdot \frac{1}{x-1}$$

Ex: Compute $\int \frac{2x^3}{3-x^4}\,dx$

$$\int \frac{2x^3}{3-x^4}\,dx \underset{du=-4x^3dx}{\overset{u=3-x^4}{=\!=}} \int \frac{1}{u}\cdot\left(\frac{-1}{2}du\right) = -\frac{1}{2}\ln|u| + C$$

$$= -\frac{1}{2}\ln|3-x^4| + C$$

Logarithmic Differentiation

To differentiate $y=f(x)$, an often useful technique is logarithmic differentiation:

1) Take ln of both sides:

$$\ln y = \ln f(x)$$

2) Take the derivative w.r.t. x:

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\ln f(x)$$

3) Isolate $\frac{dy}{dx} = f'(x)$:

$$f'(x) = \frac{dy}{dx} = y\frac{d}{dx}\left(\ln f(x)\right) = f(x)\frac{d}{dx}\left(\ln f(x)\right)$$

Ex: Differentiate $y = \sqrt[7]{\dfrac{\cos^2 x}{(x^2+1)^2}}$

$$\ln y = \ln \sqrt[7]{\frac{\cos^2 x}{(x^2+1)^2}} = \frac{1}{7} \ln \frac{\cos^2 x}{(x^2+1)^2} = \frac{1}{7}\left(\ln \cos^2 x - \ln (x^2+1)^2\right)$$

$$= \frac{1}{7}\left(2 \ln \cos x - 2 \ln (x^2+1)\right)$$

Differentiate:

$$\frac{1}{y} y' = \frac{2}{7}\left(\frac{-\sin x}{\cos x} - \frac{2x}{x^2+1}\right)$$

$$\Rightarrow y' = \frac{-2}{7} y \left(\tan x - \frac{2x}{x^2+1}\right)$$

$$= \frac{-2}{7} \sqrt[7]{\frac{\cos^2 x}{(x^2+1)^2}} \left(\tan x - \frac{2x}{x^2+1}\right)$$